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A Pseudospectral Method for the Solution of the Two-Dimensional Navier–Stokes Equations in the Primitive Variable Formulation

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A pseudospectral method was developed for the solution of the Navier–Stokes equations for incompressible flows in the primitive-variable formulation. The method employs Chebyshev expansion methods in order to generate approximations to the momentum and pressure equtions and utilizes a well-known fractional time-step procedure in obtaining a solution to these equations. Results for the buoyancy-driven flow in a square enclosure with nonisothermal vertical and insulated horizontal walls, at Rayleigh numbers in the range 1.4×10^4–1.4×10^6, are represented. The proposed method is easy to implement and can accurately reproduce the physics of the flow under consideration from transient to steady state. © 1986 Academic Press, Inc.

INTRODUCTION

Two dimensional laminar flows of incompressible Newtonian flows have been extensively studied by different numerical schemes applied to the following formulations of the Navier–Stokes equations:

(i) Vorticity-stream function.

(ii) Vorticity–velocity [1].

(iii) Primitive variables.

The vorticity-stream function and vorticity–velocity formulations share the difficulties of appropriate implementation of the boundary conditions for vorticity. On the other hand, schemes proposed for the primitive-variable formulation have difficulties with the determination of pressure in accordance with incompressibility. Pressure is determined either from a Poisson equation with Neumann boundary conditions or from the incompressibility contraint [2].

The application of spectral and pseudospectral methods to fluid flow problems is still in its growth phase with a number of unresolved problems. Recent calculations by Orszag and Kells [3], Morchoisne [4], Taylor and Murdock [5] and Kleiser and Schumann [6] have aided considerably the advances in this area.

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In a previous paper [8], pseudospectral methods were applied to the driven-cavity and pipe entrance flows, in the vorticity-stream function and vorticity–velocity formulations, respectively. Here, these methods are applied to the solution of the Navier–Stokes equations in the primitive variable formulation for a buoyancy-driven transient flow in a square enclosure.

A Pseudospectral Method for the Two-Dimensional Navier–Stokes Equations in Primitive Variables

Before proceeding with the outline of the method, we briefly discuss the calculation of derivatives at the selected points $x_n = \cos[\pi(n-1)/N]$ $(1 \leqslant n \leqslant N+1)$ from Chebyshev expansions [7].

The first and second derivatives of a function $f(x)$ with $x \in [-1, 1]$ can be approximated as

$$\mathbf{f}' = \mathbf{G}^{(1)}\mathbf{f} \tag{1a}$$

$$\mathbf{f}'' = \mathbf{G}^{(2)}\mathbf{f}, \tag{1b}$$

where

$$\mathbf{G}^{(q)} = \mathbf{T}\mathbf{G}^{(q)}\hat{\mathbf{T}}, \qquad q = 1, 2 \tag{2a}$$

and

$$\mathbf{G}^{(2)} = \mathbf{G}^{(1)}\mathbf{G}^{(1)} \tag{2b}$$

with

$$\mathbf{T} = \begin{bmatrix} T_0(x_1) & T_1(x_1) & \cdots & T_N(x_1) \\ T_0(x_2) & T_1(x_2) & \cdots & T_N(x_2) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ T_0(x_N) & T_1(x_N) & \cdots & T_N(x_N) \\ T_0(x_{N+1}) & T_1(x_{N+1}) & \cdots & T_N(x_{N+1}) \end{bmatrix} \tag{3a}$$

$$\hat{\mathbf{T}} = \frac{2}{N} \begin{bmatrix} T_0(x_1)/4 & T_0(x_2) & \cdots & T_0(x_N)/2 & T_0(x_{N+1})/4 \\ T_1(x_1)/2 & T_1(x_2) & \cdots & T_1(x_N) & T_1(x_{N+1})/2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ T_{N-1}(x_1)/2 & T_{N-1}(x_2) & \cdots & T_{N-1}(x_N) & T_{N-1}(x_{N+1})/2 \\ T_N(x_1)/4 & T_N(x_2)/2 & \cdots & T_N(x_N)/2 & T_N(x_{N+1})/4 \end{bmatrix}$$

and $\mathbf{G}^{(1)}$ being an $(N+1) * (N+1)$ matrix with elements

$$\begin{aligned} G^{(1)}_{i,j} &= 0 \qquad \text{if} \quad i \geqslant j \text{ or } i+j \text{ even} \\ &= \frac{2(j-1)}{c_i} \qquad \text{otherwise} \end{aligned} \tag{3b}$$

$(c_1 = 2,\ c_p = 1,\ p \geqslant 2)$.

We adopt Chorin's splitting technique [2], which is a first-order approximation in time, to formulate Poisson's equation for the pressure and then use a second-order modified Euler scheme to obtain increased accuracy in time.

Chorin's method involves writing the momentum equations as

$$\frac{\partial u_i}{\partial t} + \frac{\partial p}{\partial x_i} = F_i, \tag{4}$$

where

$$F_i = v\frac{\partial^2 u_i}{\partial x_j^2} - u_j\frac{\partial u_i}{\partial x_j} + S_i,$$

and S_i is a forcing term.

In the first step, assuming that u_i^n represents the discrete approximation to the solution at time $n\Delta t$, an auxiliary velocity vector u_i^A can be explicitly found from

$$u_i^A - u_i^n = \Delta t F_i. \tag{5}$$

In the second step, the flow field is corrected via the equations

$$u_i^{n+1} = u_i^A - \Delta t \frac{\partial p^{n+1}}{\partial x_i} \tag{6a}$$

$$\frac{\partial u_i^{n+1}}{\partial x_i} = 0. \tag{6b}$$

The discrete form of the Chebyshev expansion method for the second step, Eq. (6a), can be written as

$$u^{n+1}_{i,j} = u^A_{i,j} - \Delta t \sum_{m=1}^{NX+1} \hat{G}X^{(1)}_{i,m} p^{n+1}_{m,j} \tag{7a}$$

$$v^{n+1}_{i,j} = v^A_{i,j} - \Delta t \sum_{l=1}^{NY+1} \hat{G}Y^{(1)}_{j,l} p^{n+1}_{i,l}. \tag{7b}$$

By requiring that $u_{i,j}^{n+1}$ and $v_{i,j}^{n+1}$ satisfy continuity, and with no-slip boundary conditions, Eq. (6b) becomes

$$\sum_{m=1}^{NX+1} \hat{G}X_{i,m}^{(1)} u_{m,j}^{A} - \Delta t \sum_{n=2}^{NX} \hat{G}X_{i,n}^{(1)} \sum_{m=1}^{NX+1} \hat{G}X_{n,m}^{(1)} p_{m,j}^{n+1} + \sum_{l=1}^{NY+1} \hat{G}Y_{j,l}^{(1)} v_{i,l}^{A} - \Delta t \sum_{k=2}^{NY} \hat{G}Y_{j,k}^{(1)} \sum_{l=1}^{NY+1} \hat{G}Y_{k,l}^{(1)} p_{i,l}^{n+1} = 0. \tag{8}$$

With the matrix identifications

$$\mathbf{BX} = \begin{bmatrix} \hat{G}X_{12}^{(1)} & \cdots & GX_{1,NX}^{(1)} \\ .. & \cdots & .. \\ .. & \cdots & .. \\ \hat{G}X_{NX+1,2}^{(2)} & \cdots & \hat{G}X_{NX+1,NX}^{(1)} \end{bmatrix}_{(NX+1)\otimes(NX-1)} \otimes \begin{bmatrix} \hat{G}X_{21}^{(1)} & \cdots & \hat{G}X_{2,NX+1}^{(1)} \\ .. & \cdots & .. \\ .. & \cdots & .. \\ \hat{G}X_{NX,1}^{(1)} & \cdots & \hat{G}X_{NX,NX+1}^{(1)} \end{bmatrix}_{(NX-1)\otimes(NX+1)} \tag{9a}$$

and

$$\mathbf{BY} = \begin{bmatrix} \hat{G}Y_{12}^{(1)} & \cdots & GY_{1,NY}^{(1)} \\ .. & \cdots & .. \\ .. & \cdots & .. \\ \hat{G}Y_{NY+1,2}^{(1)} & \cdots & \hat{G}Y_{NY+1,NY}^{(1)} \end{bmatrix}_{(NY+1)\otimes(NY-1)} \otimes \begin{bmatrix} \hat{G}Y_{21}^{(1)} & \cdots & GY_{2,NY+1}^{(1)} \\ .. & \cdots & .. \\ .. & \cdots & .. \\ \hat{G}Y_{NY,1}^{(1)} & \cdots & \hat{G}Y_{NY,NY+1}^{(1)} \end{bmatrix}_{(NY-1)\otimes(NY+1)} \tag{9b}$$

Equation (8) can be rearranged into an equation for the pressure

$$\sum_{m=1}^{NX} BX_{i,m} p_{m,j}^{n+1} + \sum_{l=1}^{NY+1} BY_{j,l} p_{i,l}^{n+1} = \frac{1}{\Delta t}\left[\sum_{m=2}^{NX} \hat{G}X_{i,m}^{(1)} u_{m,j}^{A} + \sum_{l=2}^{NY} \hat{G}Y_{j,l}^{(1)} v_{i,l}^{A}\right] \tag{10}$$

with $i = 2,..., NX$ and $j = 2,..., NY$.

On the boundaries, $i=1$, $NX+1$ and $j=1$, $NY+1$, the conditions $\partial v/\partial y=0$ and $\partial u/\partial x=0$, respectively, are approximated by

$$\sum_{m=1}^{NY+1} BX_{i,m}\, p_{m,j}^{n+1} = \frac{1}{\Delta t} \sum_{m=2}^{NX} \hat{G}X_{i,m}^{(1)} u_{m,j}^{A}, \qquad i=1 \text{ or } NX+1 \tag{11a}$$

and

$$\sum_{l=1}^{NY+1} BY_{j,l}\, p_{i,l}^{n+1} = \frac{1}{\Delta t} \sum_{l=2}^{NY} \hat{G}Y_{j,l}^{(1)} v_{i,l}^{A}, \qquad j=1 \text{ or } NY+1. \tag{11b}$$

Equation (10) with boundary conditions given by Eq. (11a) and (11b) is the discrete form of the Poisson equation for the pressure and can be solved by the LU decomposition. Since the operator appearing in the equation for the pressure is linear it only needs to be inverted once.

A modified Euler scheme is used for the flow field. This scheme involves the following steps:

Step I

$$\bar{u}_i^A - u_i^n = \Delta t F_i^n \tag{12}$$

$$\bar{u}_i^{n+1} = \bar{u}_i^A - \Delta t \frac{\partial \bar{p}^{n+1}}{\partial x_i} \tag{13a}$$

$$\frac{\partial \bar{u}_i^{n+1}}{\partial x_i} = 0. \tag{13b}$$

Step II

$$u_i^A - u_i^n = \frac{\Delta t}{2} [\bar{F}_i^{n+1} + F_i^n] \tag{14}$$

$$u_i^{n+1} = u_i^A - \Delta t \frac{\partial p^{n+1}}{\partial x_i} \tag{15a}$$

$$\frac{\partial u_i^{n+1}}{\partial x_i} = 0. \tag{15b}$$

A Bouyancy-Driven Flow in a Square Enclosure

The method described above is applied to the problem of a buoyancy-driven flow in a square cavity with walls of length l which enclose a Newtonian incompressible fluid at temperature T_0 initially. At time $t=0$, the temperature of the left vertical wall is raised to a level T_1 and is maintained at this level thereafter. In contrast to vertical walls which are thermally conducting, the horizontal walls are assumed to be completely insulated. When the temperature of the left vertical wall is raised, a buoyancy-driven flow commences. A measure of the strength of the buoyant forces as compared to the viscous forces is provided by the Rayleigh number which is defined as

$$\mathrm{Ra} = \frac{g\beta l^3(T_1 - T_0)}{\nu\alpha}$$

(g is the gravity acceleration, β the thermal expansion coefficient, ν the kinematic viscosity and α the thermal diffusivity).

Numerical solutions to this problem are usually confined to low Rayleigh numbers. At high Rayleigh numbers a number of numerical methods are impaired by stability problems and exceedingly refined grid requirements. This is not the case for the Chebyshev expansion methods which allow accurate resolution of thin boundary layers associated with high Rayleigh numbers.

When distances, velocities and time are made dimensionless by dividing them by l, ν/l and $l^2/\nu\,\mathrm{Ra}$, respectively, and a dimensionless temperature is defined as $(T - T_0)/(T_1 - T_0)$, the governing equations, with the Bousssinesq approximation, become

$$\frac{\partial u}{\partial t} + \frac{1}{\mathrm{Ra}}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial p}{\partial x} + \frac{1}{\mathrm{Ra}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{16a}$$

$$\frac{\partial v}{\partial t} + \frac{1}{\mathrm{Ra}}\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Ra}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{T}{\mathrm{Pr}} \tag{16b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{16c}$$

$$\frac{\partial T}{\partial t} + \frac{1}{\mathrm{Ra}}\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{1}{\mathrm{Ra\,Pr}}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial T}{\partial y^2}\right). \tag{16d}$$

The initial and boundary conditions are given by

$$t = 0; \quad 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1, \quad u = v = 0;\ T = 0 \tag{17}$$

$$t > 0; \quad x = 0 \text{ and } 1,\ y = 0 \text{ and } 1, \quad u = v = 0 \tag{18a}$$

$$x = 0, \quad T = 1 \tag{18b}$$

$$x = 1, \quad T = 0 \tag{18c}$$

$$y = 0 \text{ and } 1, \quad \partial T/\partial y = 0 \tag{18d}$$

The discretized forms of Eq. (16)–(18) together with the Poisson equation for the pressure are solved using the fractional time-step pseudospectral method described earlier. The boundary conditions of Eq. (18d) are approximated by

$$\sum_{l=1}^{NY+1} \hat{G}Y^{(1)}_{1,l} T_{i,l} = 0, \qquad i = 2, ..., NX \tag{19a}$$

$$\sum_{l=1}^{NY+1} \hat{G}Y^{(1)}_{NY+1,l} T_{i,l} = 0, \qquad i = 2, ..., NX, \tag{19b}$$

which constitute the additionally required equations for $T_{i,1}$ and $T_{i,NY+1}$.

Numerical calculations were carried out for Pr = 7, the Prandtl number for water, and three different Rayleigh numbers: 1.4×10^4, 1.4×10^5 and 1.4×10^6. The time steps for each case were initially chosen to be $\Delta t = 1$, 10 and 100 and were gradually increased to $\Delta t = 3$, 30 and 250, respectively. Throughout the computations isotherms were generated by Chebyshev polynomial interpolation [12]. According to this interpolation, the function at an intermediate point x is calculated from the values of the function at the collocation points $\tilde{x}_i$, $i = 1, ..., N+1$, through

$$f(x) = \mathbf{T}^T(x)\,\mathbf{a} \approx \mathbf{T}^T(x)\,\hat{\mathbf{T}}\mathbf{f}(\tilde{\mathbf{x}}) \tag{20}$$

where $\mathbf{T}^T(x) = [T_0(x), T_1(x), ..., T_{N+1}(x)]$; $\tilde{\mathbf{x}} = [\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_{N+1}]$; $\mathbf{a} = [a_1, a_2, ..., a_{N+1}]$ and a_p, $p = 1, ..., N+1$, are the coefficients in the expansion

$$f(x_n) = \sum_{p=1}^{N+1} a_p T_p(x_n), \qquad n = 1, ..., N+1.$$

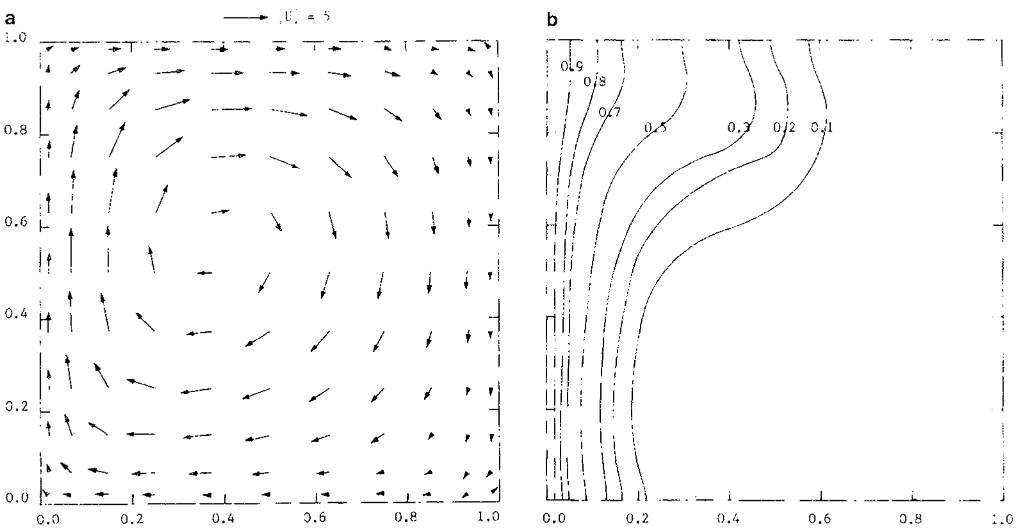

FIG. 1. (a) Flow direction vectors at time $t/\mathrm{Ra} = 0.15$, with $\mathrm{Ra} = 1.4 \times 10^4$, Pr = 7. (b) Isotherms at time $t/\mathrm{Ra} = 0.15$, with $\mathrm{Ra} = 1.4 \times 10^4$, Pr = 7.

Flow patterns and isotherms for all three different cases are shown in Fig. 1–5. The discussion below focuses on the development of the flow and temperature fields for $\mathrm{Ra} = 1.4 \times 10^4$ from transient to steady state. At $t/\mathrm{Ra} = 0.15$, the flow (Fig. 1a) is initiated near the upper part of the left vertical wall. Changes in temperature (Fig. 1b) also originate here. This region is constantly supplied with warm fluid while the lower part is continuously supplied with cool fluid and displacement of the boundary layer is suppressed. At time $t/\mathrm{Ra} = 0.707$ (Figs. 2a, 2b), the flow and temperature fields near the top of the cavity reach steady state, while the bottom region still continues to develop. At time $t/\mathrm{Ra} = 2$, both flow and temperature fields have reached steady state everywhere in the cavity.

The fully developed flow and temperature fields for $\mathrm{Ra} = 1.4 \times 10^5$ at time $t/\mathrm{Ra} = 1.3$ are shown in Figs. 3a, b, respectively. The computations demonstrate the existence of two distinct circulation cells, in agreement with previous investigations [9–11]. At the higher Rayleigh number $\mathrm{Ra} = 1.4 \times 10^6$, thin boundary layers with steep changes in velocity and temperature are observed. The steady state flow and temperature fields shown in Figs. 4a, b compare will to Figs. 2a, b of Ref. [10], respectively.

An important quantity in this problem is the dimensionless heat transfer coefficient or Nusselt number defined by

$$Nu_l = \frac{hl}{k} = \frac{\partial T}{\partial x}\bigg|_{x=0} \tag{21}$$

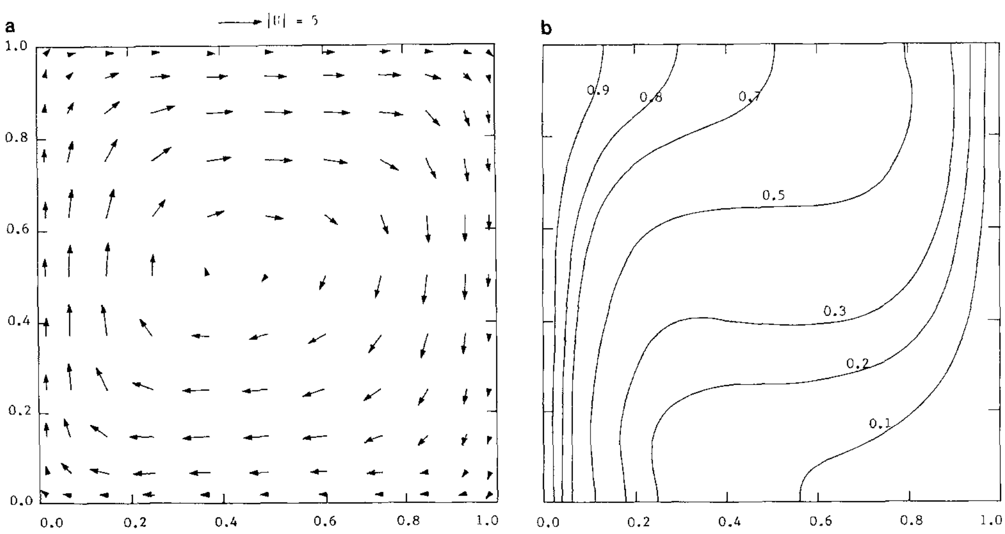

FIG. 2. (a) Flow direction vectors at time $t/\mathrm{Ra} = 0.707$, with $\mathrm{Ra} = 1.4 \times 10^4$, $\mathrm{Pr} = 7$. (b) Isotherms at time $t/\mathrm{Ra} = 0.707$, with $\mathrm{Ra} = 1.4 \times 10^4$, $\mathrm{Pr} = 7$.

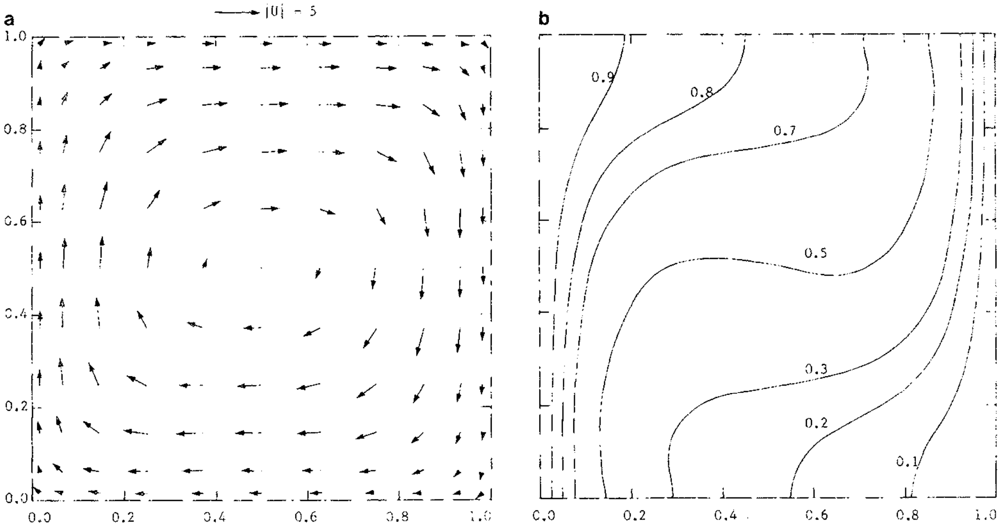

FIG. 3. (a) Flow direction vectors at steady state, with $Ra = 1.4 \times 10^4$, $Pr = 7$. (b) Isotherms at steady state, with $Ra = 1.4 \times 10^4$, $Pr = 7$.

(h is the heat transfer coefficient and k the thermal conductivity). An average Nusselt number can also defined as follows:

$$Nu = \frac{\langle h \rangle l}{k} = \int_0^1 \left.\frac{\partial T}{\partial x}\right|_{x=0} dy. \tag{22}$$

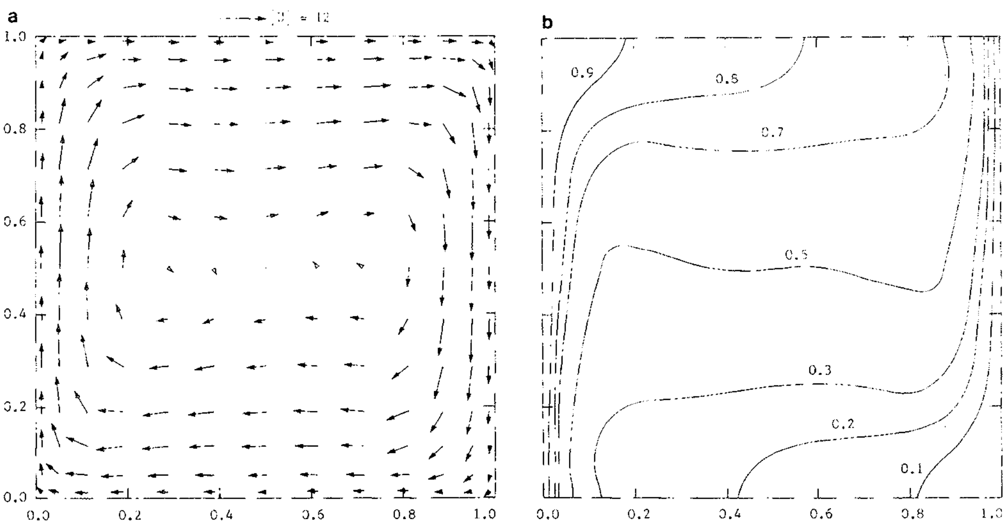

FIG. 4. (a) Flow direction vectors at steady state, with $Ra = 1.4 \times 10^5$, $Pr = 7$. (b) Isotherms at steady state, with $Ra = 1.4 \times 10^5$, $Pr = 7$.

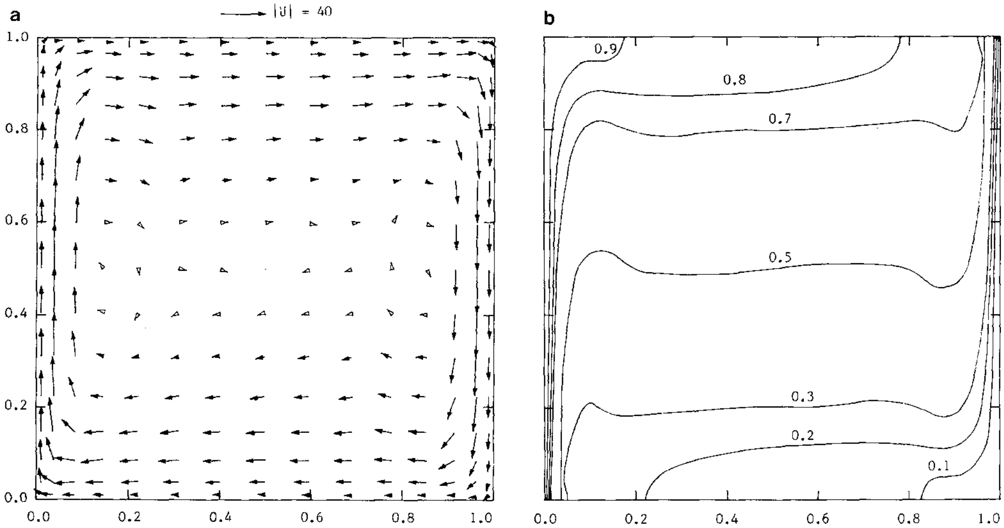

FIG. 5. (a) Flow direction vectors at steady state, with $Ra = 1.4 \times 10^6$, $Pr = 7$. (b) Isotherms at steady state, with $Ra = 1.4 \times 10^6$, $Pr = 7$.

The integral in the above expression can be approximated from Chebyshev expansions [7] as

$$Nu = \sum_{j=1}^{NY+1} W_{1,j} \sum_{m=1}^{NX+1} \hat{G}X^{(1)}_{NX+1,m} T_{m,j}, \tag{23}$$

where $W_{1,j}$ is a weighting factor.

Average Nusselt numbers obtained in this work are compared to those from finite difference computations [11] in Table I.

TABLE I

Average Nusselt Numbers for Transient Natural Convection in a Square Cavity

Ra		Mesh	Nu
10^4	[11]	65×65	2.250
1.4×10^4	present	13×13	2.546
10^5	[11]	65×65	4.573
1.4×10^5	present	15×15	5.200
10^6	[11]	65×65	9.270
1.4×10^6	present	17×17	9.834

CONCLUSIONS

A pseudospectral method which employs Chebyshev expansions and utilizes a fractional time procedure was developed for the solution of the Navier–Stokes equations in their primitive variable formulation. This method achieves accurate determination of the pressure from a Poisson-type equation and allows for overall enforcement of the compressibility constraint.

The method was applied to the solution of the problem of the buoyancy-driven flow in a square cavity with nonisothermal vertical and insulating horizontal walls. It was shown to be effective in reproducing the development of the flow and temperature fields from transient to steady state and in resolving steep changes in the field variables within thin boundary layers.

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